CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2018 Trial Examination

FORM VI

MATHEMATICS 2 UNIT

Friday 10th August 2018

General Instructions

- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 100 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 90 Marks

- Questions 11–16 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

Checklist

- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Reference Sheet
- Candidature — 102 boys

Examiner

LRP

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

What is the value of $3\cos\dfrac{\pi}{5}$, correct to three significant figures?

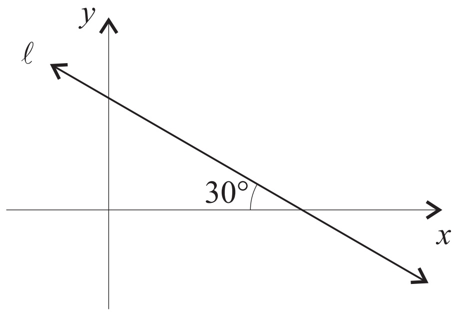
(A) $2{\cdot}42$

(B) $2{\cdot}43$

(C) $2{\cdot}99$

(D) $3{\cdot}00$

QUESTION TWO



The diagram shows the line ℓ. What is the gradient of the line ℓ?

(A) $-\sqrt{3}$

(B) $-\dfrac{1}{\sqrt{3}}$

(C) $\dfrac{1}{\sqrt{3}}$

(D) $\sqrt{3}$

QUESTION THREE

What is the maximum value of the function $y = 4 - |2\sin x|$?

(A) 0

(B) 2

(C) 4

(D) 6

Examination continues next page ...

QUESTION FOUR

Suppose $a = \log_c 2$ and $b = \log_c 3$ for constant $c > 0$. Which expression is equivalent to $\log_c 24$?

(A) $3ab$

(B) $3a + b$

(C) $a^3 b$

(D) $a^3 + b$

QUESTION FIVE

How many terms are there in the geometric sequence $2, 6, 18, \ldots, 1\,062\,882$?

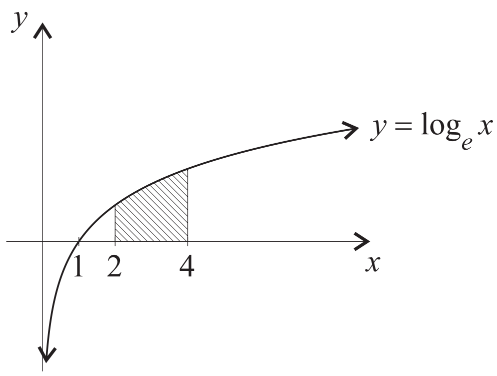
(A) 10

(B) 11

(C) 12

(D) 13

QUESTION SIX



The diagram shows the graph of $y = \log_e x$. Simpson's rule is used with three function values to approximate $\displaystyle\int_2^4 \log_e x\,dx$. What is the value of the approximation, correct to two decimal places?

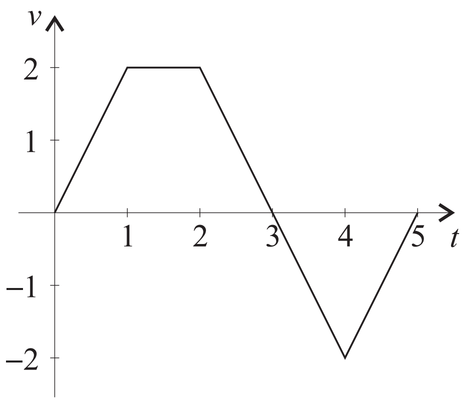
(A) 1·13

(B) 2·08

(C) 2·14

(D) 2·16

QUESTION SEVEN



A snail begins to crawl vertically up a wall at time $t = 0$. The velocity v mm/s of the snail at time t seconds, for $0 \leq t \leq 5$, is shown on the graph above.

How many seconds after it begins to crawl does the snail change direction?

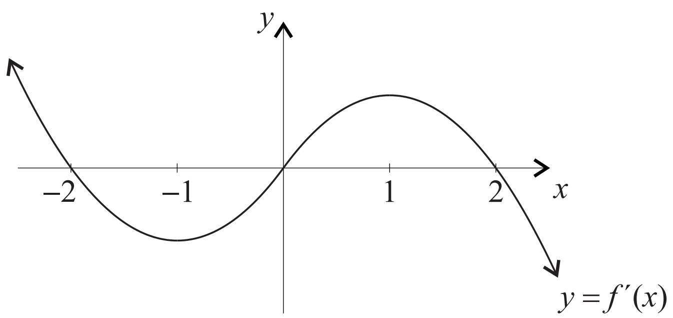
(A) 1

(B) 2

(C) 3

(D) 4

QUESTION EIGHT



The diagram shows the graph of the derivative function $y = f'(x)$.

At which value of x does a minimum turning point occur on the graph of $y = f(x)$?

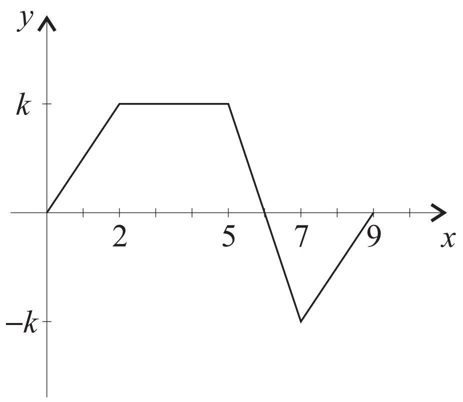
(A) -1

(B) 0

(C) 1

(D) 2

QUESTION NINE



The diagram shows the graph of $y = f(x)$. Which expression is equal to $\displaystyle\int_0^9 f(x)\,dx$?

(A) $3k$

(B) $4k$

(C) $5k$

(D) $6k$

QUESTION TEN

Given $e^y = \tan x$, for $0 < x < \frac{\pi}{2}$, which is a correct expression for $\dfrac{dy}{dx}$?

(A) $\sec x \operatorname{cosec} x$

(B) $\dfrac{1}{\tan x}$

(C) $\tan x \sec^2 x$

(D) $\sec x \tan^2 x$

End of Section I

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Factorise fully $2x^3 - 32x$. **2**

(b) Express $\dfrac{\sqrt{2}}{3-\sqrt{2}}$ with a rational denominator. **2**

(c) Find the domain of $y = \sqrt{x+8}$. **1**

(d) Differentiate:

(i) $\sin 5x$ **1**

(ii) $\dfrac{3}{x}$ **1**

(iii) $\left(e^{2x}+3\right)^4$ **2**

(e) Find:

(i) $\displaystyle\int (3x-2)^5 \; dx$ **1**

(ii) $\displaystyle\int \sec^2 7x \, dx$ **1**

(iii) $\displaystyle\int \frac{x}{x^2+1} \, dx$ **2**

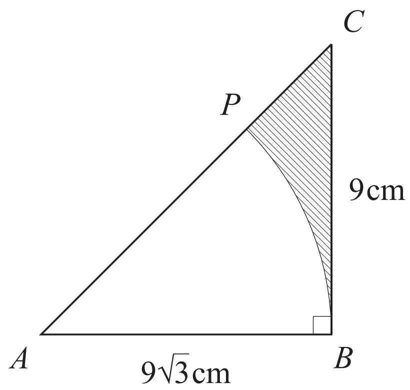
(f) Find the limiting sum of the geometric series $1 - \dfrac{1}{3} + \dfrac{1}{9} - \dfrac{1}{27} + \cdots$. **2**

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) Differentiate $x\,(1+2x)^5$. Give your answer in fully factored form. **2**

(b) The graph of $y = f(x)$ passes through the point $(-2, 1)$ and $f'(x) = 6x^2 - 5$. Find $f(x)$. **2**

(c)



The diagram shows $\triangle ABC$, where $\angle ABC = 90^\circ$, $AB = 9\sqrt{3}$ cm and $BC = 9$ cm. The circular arc BP has centre A and meets hypotenuse AC at P.

(i) Find $\angle BAC$ in exact form, in radians. **1**

(ii) Hence find the area of the shaded portion BCP, correct to the nearest square centimetre. **2**

(d) The equation $2x^2 - 5x + 1 = 0$ has roots α and β. Without finding α and β, find:

(i) $\alpha + \beta$ **1**

(ii) $\alpha\beta$ **1**

(iii) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$ **2**

(e) Consider the parabola $(x+3)^2 = -12y$.

(i) Write down the coordinates of the vertex. **1**

(ii) Find the coordinates of the focus. **1**

(iii) Write down the equation of the directrix. **1**

(iv) Sketch the parabola, showing the features found above. **1**

Examination continues overleaf ...

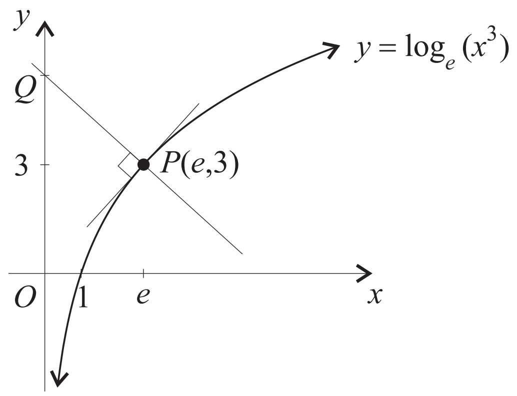
QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Consider the series $52 + 46 + 40 + 34 + \cdots$.

(i) Find a simplified expression for the sum of the first n terms. **2**

(ii) What is the maximum number of terms for which this sum remains positive? **2**

(b)



The diagram shows the graph of the function $y = \log_e(x^3)$.

(i) Find the equation of the tangent to the curve at the point $P\,(e, 3)$. **2**

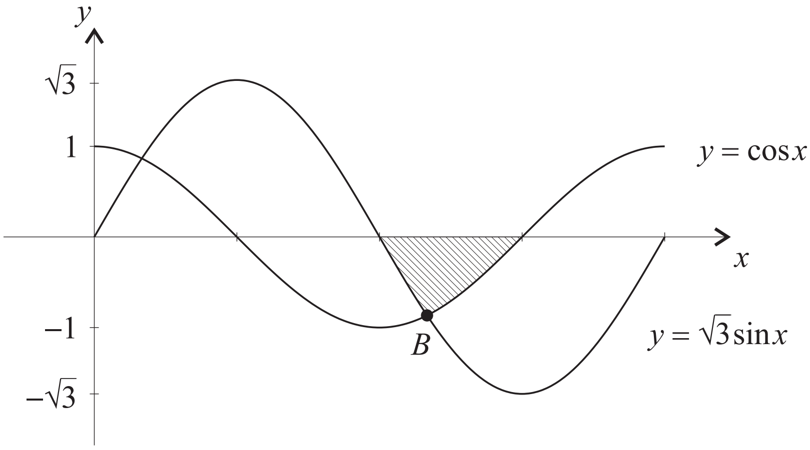
(ii) Show that the tangent at P passes through the origin O. **1**

(iii) Find the equation of the normal to the curve at P. **1**

(iv) Hence find the coordinates of the point Q where the normal meets the y-axis. **1**

(v) Hence find the area of $\triangle OPQ$ in exact form. **1**

(c)



The diagram shows the graphs of $y = \cos x$ and $y = \sqrt{3}\sin x$, for $0 \le x \le 2\pi$. The second point of intersection is labelled B.

(i) Find the x-coordinate of B. **2**

(ii) Find the area of the region shaded in the diagram, correct to three decimal places. **3**

Examination continues next page ...

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

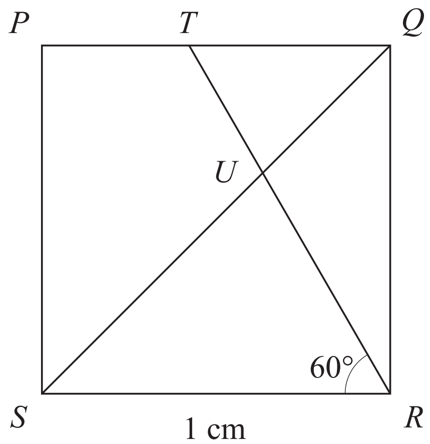
(a) Find the exact value of $\tan\theta$, given that $\sin\theta = 0{\cdot}4$ and $\cos\theta < 0$. **2**

(b) Consider the geometric series with third term 2 and eleventh term 131 072.

(i) Find the first term and common ratio. **2**

(ii) Find the fifteenth term. **1**

(c)



The diagram shows square $PQRS$, where $SR = 1\,\text{cm}$. Point T is located on side PQ, such that $\angle TRS = 60°$. The diagonal QS intersects TR at point U.

(i) Prove that $\triangle TUQ \,|||\, \triangle RUS$. **2**

(ii) Find the ratio Area $\triangle TUQ$: Area $\triangle RUS$. **2**

(d) The number of players N of a certain computer game is modelled by the equation $N = 500e^{kt}$, where k is a constant and t is the time in days since the game was first released. After two days the number of players has tripled.

(i) How many players were there at the time of the game's initial release? **1**

(ii) Find the exact value of the constant k. **2**

(iii) How many players, to the nearest player, will there be after 1 week? **2**

(iv) How many days, to the nearest day, will it take for the number of players to reach 1 000 000? **1**

QUESTION FIFTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Find $\dfrac{d}{dx}\left(2\sqrt{x}\log_e x\right)$. **2**

(ii) Hence find $\displaystyle\int \frac{\log_e x}{\sqrt{x}}\,dx$. **2**

(b) A train makes a single trip between two stations, stopping at each station. Its velocity v km/min, t minutes after leaving the first station, is given by $v = \dfrac{3t\,(6-t)}{25}$.

(i) Find the time taken to travel between the two stations. **1**

(ii) Find the maximum velocity of the train. **2**

(iii) Find the distance between the two stations. **2**

(c) At the beginning of every month, starting on the 1st of January 2019, Jack plans to deposit \$1000 into a superannuation account paying interest at a rate of 6% per annum, compounded monthly. Let $\$A_n$ be the total value of the account at the end of the nth month.

(i) Show that $A_n = 201\,000\,(1{\cdot}005^n - 1)$. **2**

(ii) If Jack keeps to his plan, what would be the total amount in his account on the 31st December 2058, that is, after 40 years? Give your answer correct to the nearest dollar. **1**

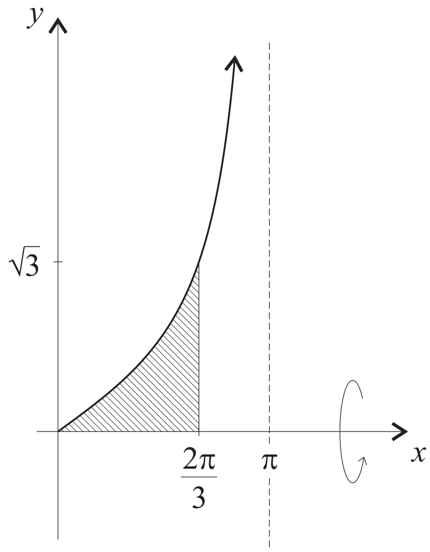
(iii) Jack has estimated that he could afford to retire once the amount in his superannuation account has reached at least \$1 600 000. Based on this, by the end of which month of what year could he first afford to retire? **2**

(iv) Jack has decided to investigate whether he could retire on 31st December 2048 with the same total of \$1 600 000, to be achieved by paying larger monthly instalments. Calculate the monthly instalment that would be required, correct to the nearest dollar. **1**

Examination continues next page ...

QUESTION SIXTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)

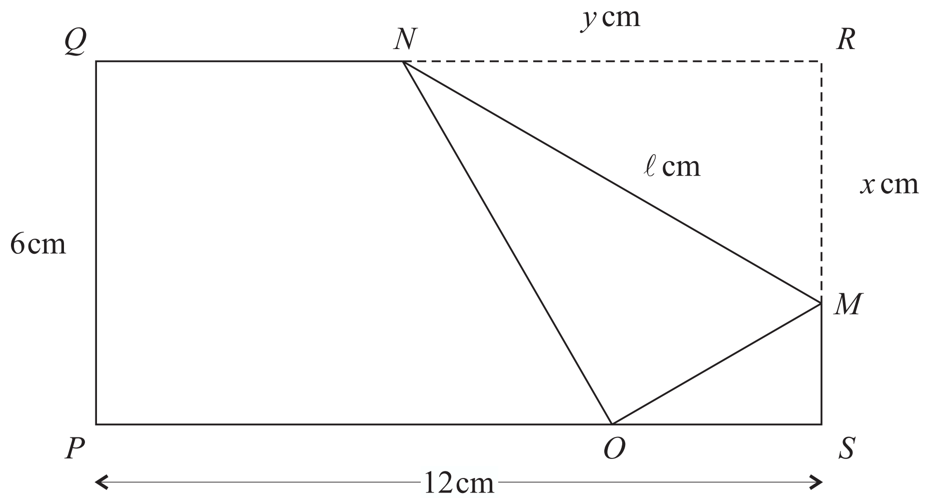


The diagram shows the region bounded by the curve $y = \tan\dfrac{x}{2}$, the x-axis and the line $x = \dfrac{2\pi}{3}$. Find the volume obtained by rotating this region about the x-axis. You will need to use the identity $1 + \tan^2\theta = \sec^2\theta$. **3**

QUESTION SIXTEEN CONTINUES ON THE NEXT PAGE

QUESTION SIXTEEN (Continued)

(b)



The diagram shows a rectangular piece of paper $PQRS$ with sides $PQ = 6$ cm and $PS = 12$ cm. The points M and N are chosen on SR and QR respectively, so that when the paper is folded along MN, the corner that was at R lands on edge PS at O.

Let $MR = x$ cm, $NR = y$ cm and $MN = \ell$ cm.

Copy or trace the diagram into your answer booklet.

(i) Show that $OS = 2\sqrt{3(x-3)}$. **2**

(ii) By considering the area of $PQRS$ as a sum of its parts, show that $y = \dfrac{x\sqrt{3}}{\sqrt{x-3}}$. **2**

(iii) You may assume that the minimum value of x occurs when point N coincides with point Q. Show that $24 - 12\sqrt{3} \le x \le 6$. **3**

(iv) Show that the crease length ℓ is given by $\ell = \sqrt{\dfrac{x^3}{x-3}}$. **1**

(v) Find the minimum possible crease length ℓ. You must justify that it is a minimum. Give your answer correct to the nearest millimetre. **4**

End of Section II

END OF EXAMINATION

MATHEMATICS 2 UNIT - TRIAL

Q1. $2 \cdot 43$ — B

Q2. $\tan 150^\circ = -\frac{1}{\sqrt{3}}$ — B

Q3. $0 \leq |2\sin x| \leq 2$

$\therefore y_{max} = 4 - 0$

$= 4$ — C

Q4. $\log_c 24 = \log_c(2^3 \times 3)$

$= 3\log_c 2 + \log_c 3$

$= 3a + b$ — B

Q5. $2 \times 3^{n-1} = 1\,062\,882$

$3^{n-1} = 531\,441$

$n = \frac{\log 531441}{\log 3} + 1$

$= 13$ — D

Q6. $\frac{4-2}{6}\left[\ln 2 + 4\ln 3 + \ln 4\right] = 2 \cdot 15796\ldots$

$\doteqdot 2 \cdot 16$ — D

Q7. 3 seconds (velocity changes from positive to negative) — C

Q8. $-\backslash \underset{0}{\_} /+$ $\therefore$ at $x = 0$ — B

Q9. the triangles cancel out

$\therefore \int_0^9 f(x)\,dx = (5-2) \times k$

$= 3k$ — A

Q10. $y = \log_e \tan x$

$\frac{dy}{dx} = \frac{1}{\tan x} \times \sec^2 x$

$= \frac{\cos x}{\sin x} \times \frac{1}{\cos^2 x}$

$= \operatorname{cosec} x \sec x$ — A

QUESTION 11:

a) $2x^3 - 32x = 2x(x^2-16)$ ✓

$= 2x(x+4)(x-4)$ ✓

b) $\frac{\sqrt{2}}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}}$ ✓ $= \frac{3\sqrt{2}+2}{9-2}$

$= \frac{3\sqrt{2}+2}{7}$ ✓

c) $x + 8 \geq 0$

$\therefore x \geq -8$ ✓

d) i) $\frac{d}{dx}(\sin 5x) = 5\cos 5x$ ✓

ii) $\frac{d}{dx}\left(\frac{3}{x}\right) = \frac{d}{dx}(3x^{-1})$

$= -3x^{-2}$ ✓

$= -\frac{3}{x^2}$

iii) $\frac{d}{dx}(e^{2x}+3)^4 = 4(e^{2x}+3)^3 \times 2e^{2x}$ ✓

$= 8e^{2x}(e^{2x}+3)^3$ ✓

e) i) $\int (3x-2)^5\,dx = \frac{(3x-2)^6}{6\times 3} + C$

$= \frac{(3x-2)^6}{18} + C$ ✓

ii) $\int \sec^2 7x\,dx = \frac{\tan 7x}{7} + C$ ✓

iii) $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx$ ✓

$= \frac{1}{2}\ln(x^2+1) + C$ ✓

f) $S_\infty = \frac{1}{1-(-\frac{1}{3})}$ ✓

$= \frac{3}{4}$ ✓

QUESTION 12:

a) Let $u = x$ $\qquad v = (1+2x)^5$

$u' = 1$ $\qquad v' = 5(1+2x)^4 \times 2$

$= 10(1+2x)^4$ ✓

$$\frac{d}{dx}\left(x(1+2x)^5\right) = (1+2x)^5 \times 1 + x \times 10(1+2x)^4$$
$$= (1+2x)^4(1+2x+10x)$$
$$= (1+2x)^4(1+12x)$$ ✓

b) $f(x) = \frac{6x^3}{3} - 5x + c$

$= 2x^3 - 5x + c$ ✓

$1 = 2(-2)^3 - 5(-2) + c$

$\therefore c = 7$

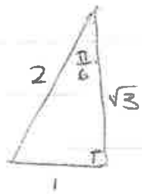
$f(x) = 2x^3 - 5x + 7$ ✓

c) i) $\tan(\angle BAC) = \frac{9}{9\sqrt{3}}$

$= \frac{1}{\sqrt{3}}$

$\therefore \angle BAC = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{6}$ ✓



ii) Area BCP $= \frac{1}{2} \times 9\sqrt{3} \times 9 - \frac{1}{2} \times (9\sqrt{3})^2 \times \frac{\pi}{6}$ ✓

$= 6.5308\ldots$

$\doteqdot 7\text{ cm}^2$ ✓ (ROUNDING)

d) i) $\alpha + \beta = -\frac{(-5)}{2}$

$= \frac{5}{2}$ ✓

ii) $\alpha\beta = \frac{1}{2}$ ✓

iii) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2 + \beta^2}{\alpha\beta}$ ✓

$= \dfrac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$

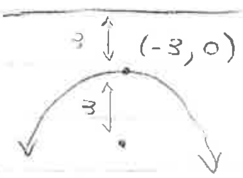
$= \dfrac{\left(\frac{5}{2}\right)^2 - 2\left(\frac{1}{2}\right)}{\frac{1}{2}}$

$= \dfrac{21}{2}$ or $10\frac{1}{2}$ ✓

e) i) $V(-3, 0)$ ✓

ii) $4a = 12$

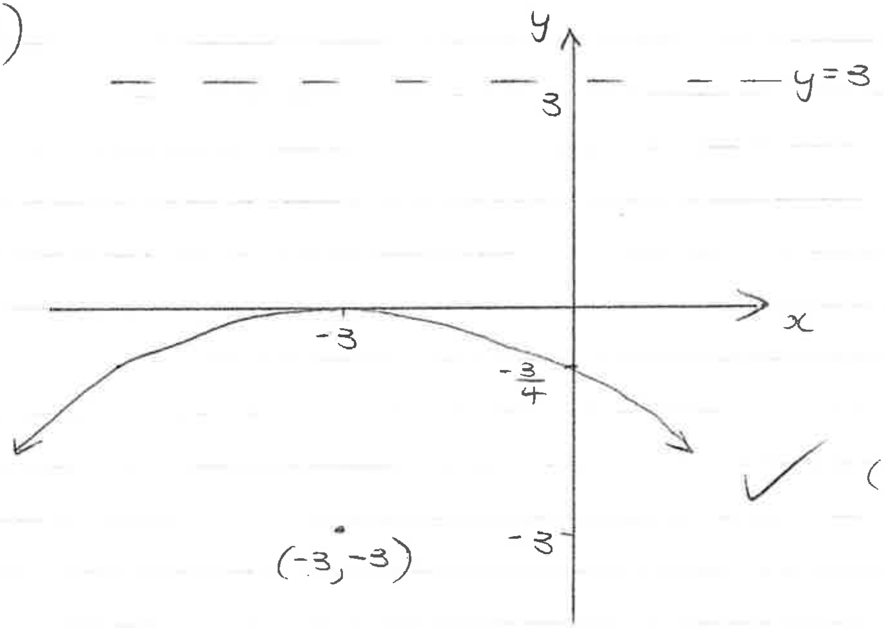
$a = 3$



Focus: $(-3, -3)$ ✓

iii) Directrix: $y = 3$ ✓

iv)



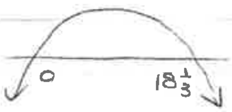
✓ (don't penalise missing y-int).

QUESTION 13:

a) i) $a = 52$
$d = -6$

$S_n = \frac{n}{2}\left[2 \times 52 + (n-1) \times -6\right]$ ✓
$= n(52 - 3n + 3)$
$= n(55 - 3n)$ ✓

ii) $n(55 - 3n) > 0$



✓

$0 < n < 18\frac{1}{3}$

$\therefore$ maximum of 18 terms ✓

b) i) $y = \log_e x^3$
$= 3\log_e x$

$\frac{dy}{dx} = \frac{3}{x}$ ✓

$y - 3 = \frac{3}{e}(x - e)$ ✓

$\therefore y = \frac{3}{e}x$

ii) when $x = 0$, $y = \frac{3}{e} \times 0$
$= 0$ ✓ $\therefore$ passes through $(0,0)$

iii) $m_\perp = -\frac{e}{3}$

$y - 3 = -\frac{e}{3}(x - e)$ ✓

$\therefore y = -\frac{e}{3}x + \frac{e^2}{3} + 3$

iv) when $x = 0$, $y = \frac{e^2}{3} + 3$
$= \frac{e^2 + 9}{3}$

$\therefore Q\left(0, \frac{e^2+9}{3}\right)$ ✓

v) $\text{Area} = \frac{1}{2} \times \left(\frac{e^2+9}{3} \right) \times e$ ✓

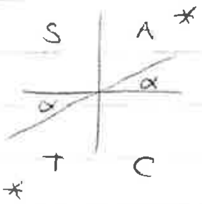
$= \frac{e(e^2+9)}{6}$ square units

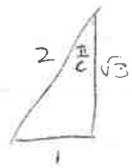
c) i) $\cos x = \sqrt{3} \sin x$

$\frac{\sin x}{\cos x} = \frac{1}{\sqrt{3}}$

$\tan x = \frac{1}{\sqrt{3}}$ ✓

$\alpha = \frac{\pi}{6}$





$x_B = \pi + \frac{\pi}{6}$

$= \frac{7\pi}{6}$ ✓

ii) $\text{Area} = \left| \int_{\pi}^{\frac{7\pi}{6}} \sqrt{3} \sin x \, dx + \int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} \cos x \, dx \right|$ ✓

$= \left| \left[-\sqrt{3} \cos x \right]_{\pi}^{\frac{7\pi}{6}} + \left[\sin x \right]_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} \right|$ ✓

$= \left| -\sqrt{3}\left(\cos \frac{7\pi}{6} - (-1)\right) + -1 - \sin \frac{7\pi}{6} \right|$

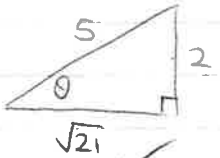
$= \left| -0.73205.. \right|$

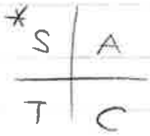
$\doteqdot 0.732$ square units ✓

QUESTION 14:

a) $\sin\theta = 0.4$

$= \frac{2}{5}$





✓

$\tan\theta = -\frac{2}{\sqrt{21}}$ ✓

b) i) $ar^2 = 2$ — ①

$ar^{10} = 131\,072$ — ②

② ÷ ①: $r^8 = 65\,536$

$\therefore r = 4$ or -4 ✓ (BOTH + & −)

$a = \frac{2}{16}$

$= \frac{1}{8}$ ✓

ii) $T_{15} = \frac{1}{8}(4)^{15-1}$ $\left(= \frac{1}{8}(-4)^{14}\right)$

$= 33\,554\,432$ ✓

c) i) In $\triangle TUQ$ & $\triangle RUS$:

$\angle TQU = \angle RSU$ (alternate angles, $PQ \parallel SR$ - opposite sides of square) ✓

$\angle TUQ = \angle RUS$ (vertically opposite)

$\therefore \triangle TUQ \,|||\, \triangle RUS$ (equiangular or AA) ✓

ii) $\angle QRT = 30°$ (adjacent angles in a right angle)

$QR = 1$ (side of square)

$\tan 30° = \frac{TQ}{1}$

$\therefore TQ = \frac{1}{\sqrt{3}}$ ✓

$TQ : RS = \frac{1}{\sqrt{3}} : 1$

Area $\triangle TUQ$: Area $\triangle RUS = \left(\frac{1}{\sqrt{3}}\right)^2 : 1^2$ (areas of similar figures)

$= 1 : 3$ ✓

[* see overpage for alternative]

d) i) when $t = 0$

$N = 500e^0$

$= 500$ ✓ $\therefore$ 500 players at the initial release

ii) when $t = 2$

$N = 1500$

$1500 = 500e^{2k}$ ✓

$e^{2k} = 3$

$\therefore k = \frac{\log_e 3}{2}$ ✓

iii) when $t = 7$

$N = ?$

$N = 500e^{\frac{\log_e 3}{2} \times 7}$ ✓

$= 23\,382 \cdot 6859\ldots$

$\doteqdot 23\,383$ players ✓ (accept $23\,382$)

iv) $t = ?$ when $N = 1\,000\,000$

$1\,000\,000 = 500e^{\frac{\log_e 3}{2}t}$

$e^{\frac{\log_e 3}{2}t} = 2000$

$t = \log_e 2000 \times \frac{2}{\log_e 3}$

$= 13 \cdot 8372\ldots$

$\doteqdot 14$ days ✓

Q14c) ii) * Alternative :

$$\tan 30^\circ = \frac{TQ}{1}$$

$$\therefore TQ = \frac{1}{\sqrt{3}}$$ ✓

$$\frac{TQ}{RS} = \frac{TU}{RU}$$ (matching sides in similar triangles)

$$\therefore TQ = \frac{TU}{RU}$$ (RS = 1, given) *

Area ΔTUQ : Area ΔRUS

$$\cancel{\tfrac{1}{2}} \times TU \times TQ \times \cancel{\sin 60^\circ} : \cancel{\tfrac{1}{2}} \times RU \times RS \times \cancel{\sin 60^\circ}$$

$$\frac{TU \times TQ}{RU \times RS} : 1$$

$$\frac{TQ \times TQ}{1} : 1$$ (from *)

✓

$$\left(\frac{1}{\sqrt{3}}\right)^2 : 1$$

$$\frac{1}{3} : 1 \quad \text{or} \quad 1 : 3$$

QUESTION 15:

a) i) Let $u = 2x^{\frac{1}{2}}$ $\quad v = \ln x$

$u' = x^{-\frac{1}{2}}$ $\quad v' = \frac{1}{x}$ ✓

$$\frac{d}{dx}\left(2\sqrt{x}\ln x\right) = \ln x \times \frac{1}{\sqrt{x}} + \frac{2\sqrt{x}}{x}$$ ✓

ii) $$\int \frac{\ln x}{\sqrt{x}}\,dx + \int \frac{2}{\sqrt{x}}\,dx = 2\sqrt{x}\ln x + C_1$$

$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\ln x - \int 2x^{-\frac{1}{2}}\,dx + C_1$$ ✓

$$= 2\sqrt{x}\ln x - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + C_2$$

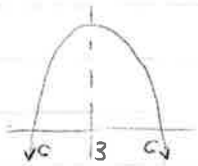
$$= 2\sqrt{x}\ln x - 4\sqrt{x} + C_2$$ ✓

b) i) $t = ?$ when $v = 0$:

$$\frac{3t(6-t)}{25} = 0$$

$\therefore t = 0$ or 6 $\quad \therefore$ it takes 6 minutes ✓

ii)



concave down parab.
∴ max at vertex

v_{max} occurs when $t = \frac{0+6}{2}$

$= 3$ ✓

$$v_{max} = \frac{3 \times 3(6-3)}{25}$$

$$= \frac{27}{25}$$

$$= 1.08 \text{ km/min}$$ ✓

iii) $$x = \int_0^6 \frac{3t(6-t)}{25}\,dt$$ ✓

$$= \frac{3}{25}\int (6t - t^2)\,dt$$

$$= \frac{3}{25}\left[3t^2 - \frac{t^3}{3}\right]_0^6$$

$$= \frac{3}{25}\left(3 \times 6^2 - \frac{6^3}{3} - (0-0)\right)$$

$$= 4.32 \text{ km}$$ ✓

c) i) 6% p.a. $= 0.5\%$ per month
$= 0.005$

1^{st} inst. invested for n months amounts to $1000(1.005)^n$
2^{nd} " $(n-1)$ " $1000(1.005)^{n-1}$
3^{rd} " $(n-2)$ " $1000(1.005)^{n-2}$
⋮
last " 1 month " $1000(1.005)$

$$A_n = 1000\left(\underbrace{1.005 + \ldots + 1.005^{n-2} + 1.005^{n-1} + 1.005^n}_{\text{G.P. } a=1.005,\ r=1.005,\ n \text{ terms}}\right)$$

$$= \frac{1000 \times 1.005(1.005^n - 1)}{1.005 - 1}$$

$$= 201\,000(1.005^n - 1)$$

OR ALTERNATIVE

$A_1 = 1000(1.005)$
$A_2 = (A_1 + 1000)1.005$
$= 1000(1.005)^2 + 1000(1.005)$
$A_3 = (A_2 + 1000)1.005$
$= 1000(1.005)^3 + 1000(1.005)^2 + 1000(1.005)$
$= 1000(1.005^3 + 1.005^2 + 1.005)$
⋮

$$A_n = 1000\left(\underbrace{1.005^n + 1.005^{n-1} + \ldots + 1.005^3 + 1.005^2 + 1.005}_{\text{G.P. } a=1.005,\ r=1.005,\ n \text{ terms}}\right)$$

$$= \frac{1000 \times 1.005(1.005^n - 1)}{1.005 - 1}$$

$$= 201\,000(1.005^n - 1)$$

ii) $n = 40 \times 12$
$= 480$

$A_{480} = 201\,000(1.005^{480} - 1)$
$= 2\,001\,448.188\ldots$
$\doteqdot \$2\,001\,448$ ✓

iii) $201\,000(1.005^{n} - 1) \geq 1\,600\,000$

$$1.005^{n} \geq \frac{1600}{201} + 1$$

$$n \geq \frac{\log_e\left(\frac{1600}{201} + 1\right)}{\log_e 1.005}$$

$\therefore n \geq 439.65\ldots$

$\therefore$ he could retire after 440 months ✓
= 36 years 8 months.

$\therefore$ earliest date = end of August 2055 ✓

iv) $n = 30 \times 12$
$= 360$

$A_{360} = 1\,600\,000$

$M = ?$

$$1\,600\,000 = \frac{M \times 1.005(1.005^{360} - 1)}{1.005 - 1}$$

$$\therefore M = \frac{1\,600\,000 \times 0.005}{1.005(1.005^{360} - 1)}$$

$= 1584.8839\ldots$
$\doteqdot \$1585$ ✓

QUESTION 16:

a) $V = \pi \int_0^{\frac{2\pi}{3}} \tan^2 \frac{x}{2} \, dx$ ✓

$= \pi \int_0^{\frac{2\pi}{3}} \left(\sec^2 \frac{x}{2} - 1\right) dx$

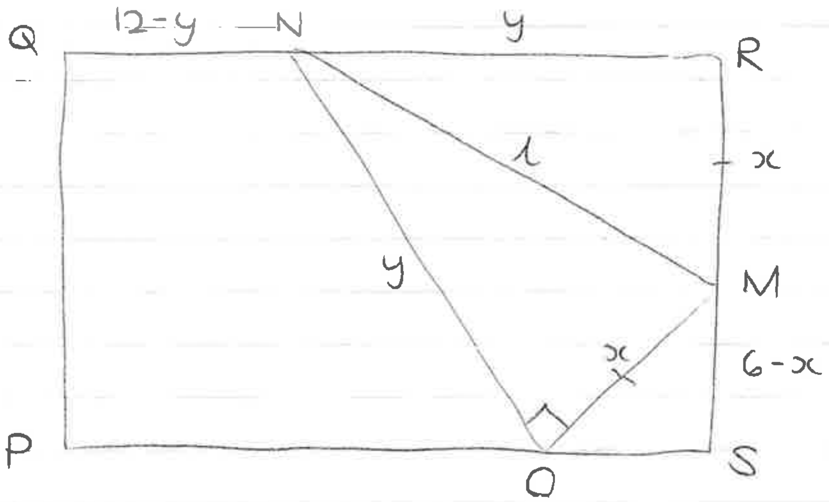
$= \pi \left[2\tan\frac{x}{2} - x \right]_0^{\frac{2\pi}{3}}$ ✓

$= \pi \left[2\tan\frac{\pi}{3} - \frac{2\pi}{3} - (0-0) \right]$

$= \pi \left(2\sqrt{3} - \frac{2\pi}{3} \right)$ ✓

$= \frac{2\pi}{3}\left(3\sqrt{3} - \pi\right)$ cubic units

b)



i) $OM = MR = x$
$SM = 6 - x$

$OS^2 = x^2 - (6-x)^2$ ✓
$= x^2 - (36 - 12x + x^2)$
$= 12(x-3)$

$\therefore OS = \sqrt{12(x-3)}$
$= 2\sqrt{3(x-3)}$

✓ show that...

ii) $|PQRS| = |PQNO| + 2|NRM| + |OMS|$

$$6\times 12 = \frac{6}{2}\left(12-y+12-2\sqrt{3(x-3)}\right)+2\left(\tfrac{1}{2}xy\right)+\tfrac{1}{2}(6-x)2\sqrt{3(x-3)} \checkmark$$

$$\cancel{72} = \cancel{72} - 3y - \cancel{6\sqrt{3(x-3)}} + xy + \cancel{6\sqrt{3(x-3)}} - x\sqrt{3(x-3)}$$

$$y(x-3) = x\sqrt{3(x-3)}$$

$$\therefore y = \frac{x\sqrt{3(x-3)}}{x-3} \checkmark$$

$$= \frac{x\sqrt{3}}{\sqrt{x-3}}$$

[* see over page for alternative]

iii) $y = 12$:

$$12 = \frac{x\sqrt{3}}{\sqrt{x-3}} \checkmark$$

$$144 = \frac{3x^2}{x-3} \quad \text{(sq. both sides)}$$

$$48(x-3) = x^2 \checkmark$$

$$x^2 - 48 + 576 = -144 + 576 \quad \text{(complete the sq.)}$$

$$(x-24)^2 = 432$$

$$\therefore x = 24 \pm \sqrt{432}$$

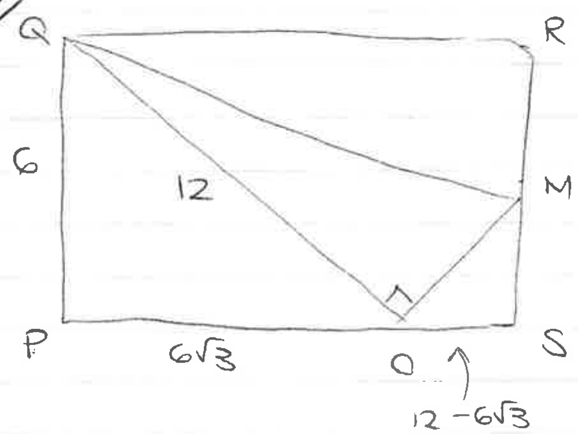
$$= 24 \pm 12\sqrt{3}$$

BUT $0 \leq x \leq 6$ (RS = 6cm) ✓

$$\therefore x_{min} = 24 - 12\sqrt{3}$$

$$\therefore 24 - 12\sqrt{3} \leq x \leq 6$$

OR//



$OQ = 12$ for x_{min}

$$\therefore OP = \sqrt{12^2 - 6^2}$$

$$= \sqrt{108}$$

$$= 6\sqrt{3}$$

$$OS = 12 - 6\sqrt{3}$$

$12 - 6\sqrt{3} = 2\sqrt{3(x-3)}$ from (i)

$$6 - 3\sqrt{3} = \sqrt{3(x-3)}$$

$$36 - 36\sqrt{3} + 27 = 3x - 9$$

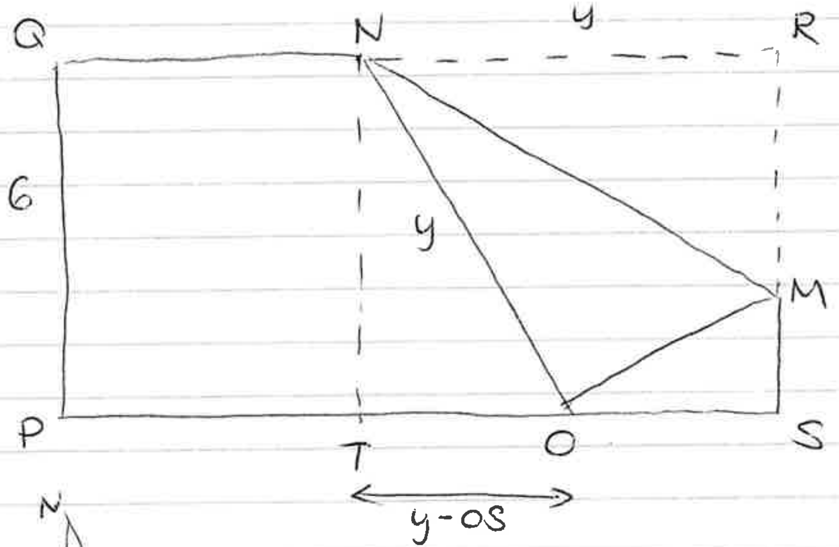
$$3x = 72 - 36\sqrt{3}$$

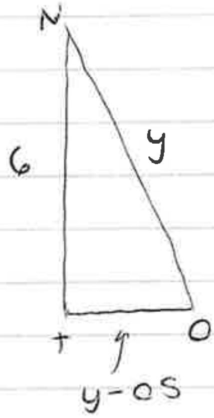
$$\therefore x_{min} = 24 - 12\sqrt{3}$$

$x_{max} = RS = 6$.

* Alternative

Q16c) ii) On this occasion, the following solution was also accepted:





$$y^2 = 6^2 + (y - OS)^2$$

$$\cancel{y^2} = 6^2 + \cancel{y^2} - 2y\,OS + OS^2$$

$$2y\,OS = 36 + OS^2$$

$$\therefore y = \frac{36 + OS^2}{2OS}$$

$$= \frac{36 + 4(3(x-3))}{2(2\sqrt{3(x-3)})}$$

$$= \frac{3x}{\sqrt{3(x-3)}}$$

$$= \frac{x\sqrt{3}}{\sqrt{x-3}}$$

iv) $l^2 = x^2 + y^2$

$$= x^2 + \left(\frac{x\sqrt{3}}{\sqrt{x-3}}\right)^2$$

$$= \frac{x^2(x-3) + 3x^2}{x-3} \quad \checkmark \quad \text{show that...}$$

$$= \frac{x^3}{x-3}$$

$$\therefore l = \sqrt{\frac{x^3}{x-3}}$$

v) $$\frac{dl}{dx} = \frac{1}{2}\left(\frac{x^3}{x-3}\right)^{-\frac{1}{2}} \times \left(\frac{(x-3)\times 3x^2 - x^3 \times 1}{(x-3)^2}\right) \checkmark$$

$$= \frac{\sqrt{x-3}}{2x\sqrt{x}} \times \frac{2x^3 - 9x^2}{(x-3)^2} \checkmark$$

$$= \frac{x^2(2x-9)}{2x\sqrt{x}(x-3)\sqrt{x-3}}$$

$$= \frac{\sqrt{x}(2x-9)}{2(x-3)\sqrt{x-3}}$$

OR $\quad l = \dfrac{x^{3/2}}{(x-3)^{\frac{1}{2}}}$ ALTERNATIVE

$$\frac{dl}{dx} = \frac{(x-3)^{\frac{1}{2}} \times \frac{3}{2}x^{\frac{1}{2}} - x^{\frac{3}{2}} \times \frac{1}{2}(x-3)^{-\frac{1}{2}}}{\left((x-3)^{\frac{1}{2}}\right)^2} \quad (\checkmark)$$

$$= \frac{\frac{3\sqrt{x(x-3)}}{2} - \frac{x\sqrt{x}}{2\sqrt{x-3}}}{x-3}$$

$$= \frac{3\sqrt{x}(x-3) - x\sqrt{x}}{2(x-3)\sqrt{x-3}} \quad (\checkmark)$$

$$= \frac{\sqrt{x}(2x-9)}{2(x-3)\sqrt{x-3}}$$

* OR look at minimising l^2 ALTERNATIVE
(see page 16)

$\frac{dl}{dx} = 0$ when $x = 0$ or $\frac{9}{2}$

<u>BUT</u> $x > 0$ $\therefore$ investigate $x = \frac{9}{2}$

x	4	4.5	5
$\frac{dl}{dx}$	-1	0	0.395...
	\	—	/

✓

$\therefore$ minimum l occurs when $x = 4.5$ cm

$\therefore$ $l_{min} = \sqrt{\frac{4.5^3}{4.5-3}}$

$= 7.7942 \ldots$

$\doteqdot 7.8$ cm (to nearest mm) ✓

(accept exact value $\sqrt{\left(\frac{9}{2}\right)^3 \times \frac{2}{3}}$

$= \sqrt{\frac{81 \times 3}{4}}$

$= \frac{9\sqrt{3}}{2}$)

* Alternative looking at l^2

Since $l > 0$, the min of l coincides with the min of l^2

$$l^2 = \frac{x^3}{x-3}$$

$$\frac{dl^2}{dx} = \frac{(x-3)\times 3x^2 - x^3 \times 1}{(x-3)^2}$$

$$= \frac{2x^3 - 9x^2}{(x-3)^2}$$

$$= \frac{x^2(2x-9)}{(x-3)^2}$$

$\frac{dl^2}{dx} = 0$ when $x = 0$ or $\frac{9}{2}$

$\underline{\underline{\text{BUT}}}$ $x > 0$ $\therefore$ investigate $x = \frac{9}{2}$

x	4	4.5	5
$\frac{dl^2}{dx}$	-16	0	6.25
	\	—	/

$\therefore$ min l^2 (& thus min l) occurs when $x = \frac{9}{2}$

$$\& \quad l_{min} = \sqrt{\frac{\left(\frac{9}{2}\right)^3}{\frac{9}{2} - 3}}$$

$$= \frac{9\sqrt{3}}{2}$$

$$= 7.7942\ldots$$

$$\doteqdot 7.8 \text{ cm}$$